Eighth-order Derivative-Free Family of Iterative Methods for Nonlinear Equations

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Abstract

In this note, we present an eighth-order derivative-free family of iterative methods for nonlinear equations. The proposed family shows optimal eight-order of convergence in the sense of the Kung and Traub conjecture [5] and is based on the Steffensen derivative approximation used in the Newton-method. As a final step, having in mind computational purposes, a derivative-free polynomial base interpolation is used in order to get optimal order of convergence with only four functional evaluations. Numerical esperiments and few issues are discussed at the end of this note.

Keywords: Non-linear equations, Steffensen's method, Polynomial interpolation, Iterative methods

1. Introduction

Let $f : D \subseteq \Re \longrightarrow \Re$ be a sufficiently differentiable function of single variable in some neighborhood D of α, where α is a simple root $(f'(\alpha) \neq 0)$ of nonlinear algebraic equation $f(x) = 0$. The well-known Newton method is denined by the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which shows a second-order convergence. One can easily get Steffensen's approximation for first order derivative as

$$\begin{cases} f(x_n - \kappa f(x_n)) & \approx f(x_n) - \kappa f(x_n) f'(x_n), \\ \kappa f(x_n) f'(x_n) & \approx f(x_n) - f(x_n - \kappa f(x_n)), \\ f'(x_n) & \approx \dfrac{1}{\kappa} \dfrac{f(x_n) - f(x_n - \kappa f(x_n))}{f(x_n)}. \end{cases} \tag{2}$$

If we substitute the derivative approximation (2) in (1), we obtain Steffensen's second order accurate derivative-free iterative method for non-linear equations [1].

$$\begin{cases} w_n & = x_n - \kappa f(x_n), \\ x_{n+1} & = x_n - \kappa \dfrac{f(x_n)^2}{f(x_n) - f(w_n)}. \end{cases} \tag{3}$$

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In 2012, an optimal eighth-order iterative method [2] was proposed by Y. Khan et al. as

$$\begin{cases} y_n &= x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - G\left(\dfrac{f(y_n)}{f(x_n)}\right)\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \dfrac{\mu}{\mu + \nu q_n^2}\dfrac{f(z_n)}{f'(z_n)}, \end{cases} \tag{4}$$

where $\mu \neq 0$, $\nu \in \Re$, $q_n = f(z_n)/f(x_n)$, $G(t)$ is a real-valued function with $G(0) = 1$, $G'(0) = 2$, $G''(0) < \infty$, and

$$\begin{cases} f'(z_n) &\approx K - C(y_n - z_n) - D(y_n - z_n)^2, \\ H &= \dfrac{f(x_n) - f(y_n)}{x_n - y_n}, \\ K &= \dfrac{f(x_n) - f(z_n)}{y_n - z_n}, \\ D &= \dfrac{f'(x_n) - H}{(x_n - y_n)(x_n - z_n)} - \dfrac{H - K}{(x_n - z_n)^2}, \\ C &= \dfrac{H - K}{x_n - z_n} - D(x_n + y_n - 2z_n). \end{cases} \tag{5}$$

In the original draft of paper [2] the expression for C has typo-mistake, which is corrected here. Actually (5) polynomial interpolation approximation for $f'(z_n)$ is given in [3]. Clearly (4) iterative scheme is not derivative free. The main contribution in this paper is to use the idea of iterative scheme (4) by introducing Steffensen's derivative approximation for $f'(x_n)$ and then finally construct derivative-free approximation for $f'(z_n)$ without reducing order of convergence.

2. Construction of derivative-free family

First we construct an interpolation polynomial approximation for $f'(z_n)$. Suppose we have $f(x_n)$, $f(w_n)$ (defined in 3), $f(y_n)$ and $f(z_n)$, One could construct a three-degree polynomial as follows

$$\begin{cases} p(\phi) &= f(y_n) + r_1(\phi - y_n) + r_2(\phi - y_n)^2 + r_3(\phi - y_n)^3, \\ p'(\phi) &= r_1 + 2r_2(\phi - y_n) + 3r_3(\phi - y_n)^2. \end{cases} \tag{6}$$

By using four functional values, we get the following system of equations:

$$\begin{cases} v_1 &= r_1 a + r_2 a^2 + r_3 a^3, \\ v_2 &= r_1 b + r_2 b^2 + r_3 b^3, \\ v_3 &= r_1 c + r_2 c^2 + r_3 c^3, \end{cases} \tag{7}$$

where

$$\begin{cases} v_1 &= f(x_n) - f(y_n), \\ v_2 &= f(z_n) - f(y_n), \\ v_3 &= f(w_n) - f(y_n), \\ a &= x_n - y_n, \\ b &= z_n - y_n \\ c &= w_n - y_n. \end{cases} \tag{8}$$

After solving (7) for r_1, r_2 and r_3 and substituting them in (6) implies the following approximation for $f'(z_n)$:

$$f'(z_n) \approx \psi_n = \frac{b(b-c)}{(a-b)(a-c)}\frac{v_1}{a} + \frac{-3b^2 + 2bc + 2ab - ac}{(a-b)(b-c)}\frac{v_2}{b} + \frac{b(b-a)}{(a-c)(b-c)}\frac{v_3}{c}. \tag{9}$$

We consider the following family of iterative methods for nonlinear equations:

$$\begin{cases} w_n &= x_n - \kappa f(x_n), \\ y_n &= x_n - \kappa \dfrac{f(x_n)^2}{f(x_n) - f(w_n)}, \\ z_n &= y_n - \kappa \dfrac{f(y_n) f(x_n)}{f(x_n) - f(w_n)} G(t_1, t_2), \\ x_{n+1} &= z_n - \dfrac{f(z_n)}{\psi_n} H(s_1, s_2), \end{cases} \tag{10}$$

where $t_1 = \dfrac{f(y_n)}{f(x_n)}$, $t_2 = \dfrac{f(y_n)}{f(w_n)}$, $s_1 = \dfrac{f(z_n)}{f(x_n)}$, $s_2 = \dfrac{f(z_n)}{f(w_n)}$ and $\kappa(\neq 0) \in \mathfrak{R}$.

3. Convergence analysis

We state the following theorem about the order of convergence of the family described in (10).

Theorem 1. *Let $f : D \subseteq \mathfrak{R} \to \mathfrak{R}$ be a sufficiently differentiable function, and $\alpha \in D$ is a simple root of $f(x) = 0$, for an open interval D. If x_0 is chosen sufficiently close to α, then the iterative scheme given in (10) converges to α. If G and H satisfy*

$$G(0,0) = 1,\ \left.\frac{\partial G}{\partial t_1}\right|_{(0,0)} = 1,\ \left.\frac{\partial G}{\partial t_2}\right|_{(0,0)} = 1,\ H(0,0) = 1,\ \left.\frac{\partial H}{\partial s_1}\right|_{(0,0)} = 0,\ \left.\frac{\partial H}{\partial s_2}\right|_{(0,0)} = 0, \tag{11}$$

and $\dfrac{\partial^2 G}{\partial t_1^2}$, $\dfrac{\partial^2 G}{\partial t_2^2}$,$\dfrac{\partial^2 G}{\partial t_1 \partial t_2}$, $\dfrac{\partial^2 H}{\partial s_1^2}$, $\dfrac{\partial^2 H}{\partial s_2^2}$,$\dfrac{\partial^2 H}{\partial s_1 \partial s_2}$ *are bounded at* $(0,0)$ *then the iterative scheme (10) shows an order of convergence at least equal to eight.*

Proof. Let the error at step n be denoted by $e_n = x_n - \alpha$ and let us define $c_1 = f'(\alpha)$ and $c_k = \dfrac{1}{k!}\dfrac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k = 2, 3, \cdots$. If we expand f around the root α and express it in terms of powers of error e_n, we obtain

$$f(x_n) = c_1(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)), \tag{12}$$

$$\begin{aligned} f(w_n) = {}& -c_1(-1 + \kappa c_1)e_n + c_1c_2(-3\kappa c_1 + 1 + \kappa^2c_1^2)e_n^2 - c_1(4\kappa c_1c_3 + 2c_2^2\kappa c_1 - 2\kappa^2c_1^2c_2^2 - c_3 - 3c_3\kappa^2c_1^2 \\ & + c_3\kappa^3c_1^3)e_n^3 + c_1(-5\kappa c_1c_4 - 5c_2\kappa c_1c_3 + 8\kappa^2c_1^2c_2c_3 + \kappa^2c_1^2c_2^3 - 3c_2\kappa^3c_1^3c_3 + c_4 + 6c_4\kappa^2c_1^2 - 4c_4\kappa^3c_1^3 \\ & + c_4\kappa^4c_1^4)e_n^4 + \cdots + O(e_n^9), \end{aligned} \tag{13}$$

$$\begin{aligned} y_n - \alpha = {}& -c_2(-1 + \kappa c_1)e_n^2 + (2c_3 - 3\kappa c_1c_3 + c_3\kappa^2c_1^2 + 2c_2^2\kappa c_1 - 2c_2^2 - \kappa^2c_1^2c_2^2)e_n^3 + (3c_4 + 10c_2\kappa c_1c_3 - 6\kappa c_1c_4 \\ & + 4c_4\kappa^2c_1^2 - c_4\kappa^3c_1^3 - 7\kappa^2c_1^2c_2c_3 - 7c_2c_3 - 5\kappa c_1c_2^3 + 2c_2\kappa^3c_1^3c_3 + 3\kappa^2c_1^2c_2^3 + 4c_2^3 - \kappa^3c_1^3c_2^3)e_n^4 \\ & + \cdots + O(e_n^9), \end{aligned} \tag{14}$$

$$\begin{aligned} f(y_n) = {}& -c_1c_2(-1 + \kappa c_1)e_n^2 - c_1(-2c_3 + 3\kappa c_1c_3 - c_3\kappa^2c_1^2 - 2c_2^2\kappa c_1 + 2c_2^2 + \kappa^2c_1^2c_2^2)e_n^3 - c_1(-3c_4 - 10c_2\kappa c_1c_3 \\ & + 6\kappa c_1c_4 - 4c_4\kappa^2c_1^2 + c_4\kappa^3c_1^3 + 7\kappa^2c_1^2c_2c_3 + 7c_2c_3 + 7\kappa c_1c_2^3 - 2c_2\kappa^3c_1^3c_3 - 4\kappa^2c_1^2c_2^3 - 5c_2^3 \\ & + \kappa^3c_1^3c_2^3)e_n^4 + \cdots + O(e_n^9), \end{aligned} \tag{15}$$

$$\frac{f(y_n)}{f(x_n)} = -c_2(-1 + \kappa c_1)e_n + (2c_3 - 3\kappa c_1c_3 + c_3\kappa^2c_1^2 + 3c_2^2\kappa c_1 - 3c_2^2 - \kappa^2c_1^2c_2^2)e_n^2 + \cdots + O(e_n^9), \tag{16}$$

$$\frac{f(y_n)}{f(w_n)} = c_2e_n + (-\kappa c_1c_3 + 2c_2^2\kappa c_1 + 2c_3 - 3c_2^2)e_n^2 + \cdots + O(e_n^9). \tag{17}$$

The Taylor series expansion of $G(t_1,t_2)$ is given by

$$G\left(\frac{f(y_n)}{f(x_n)},\frac{f(y_n)}{f(w_n)}\right)=1+\frac{f(y_n)}{f(x_n)}+\frac{f(y_n)}{f(w_n)}+A_1\left(\frac{f(y_n)}{f(x_n)}\right)^2+A_2\left(\frac{f(y_n)}{f(w_n)}\right)^2+A_3\left(\frac{f(y_n)}{f(x_n)}\right)\left(\frac{f(y_n)}{f(w_n)}\right)+O\left(t_1^3,t_2^3\right). \tag{18}$$

By using (12), (13), (15), (16), (17), we find

$$\begin{aligned}z_n-\alpha &= (-A_1c_2^2+6\kappa^2c_1^2c_2^2-c_3+2\kappa c_1c_3-10c_2^2\kappa c_1-c_3\kappa^2c_1^2+5c_2^2+A_1c_2^2\kappa c_1+2c_2^2A_3\kappa c_1-c_2^2A_3\kappa^2c_1^2\\
&+3A_2c_2^2\kappa c_1-3A_2c_2^2\kappa^2c_1^2+A_2c_2^2\kappa^3c_1^3-A_2c_2^2-\kappa^3c_1^3c_2^2-c_2^2A_3)c_2e_n^4+(-4\kappa^2c_1^2c_3^2-4\kappa^2c_1^2c_4c_2\\
&-31A_2c_2^4\kappa c_1+36A_2c_2^4\kappa^2c_1^2-19A_2c_2^4\kappa^3c_1^3+4A_2c_2^4\kappa^4c_1^4-23c_2^4A_3\kappa c_1+18c_2^4A_3\kappa^2c_1^2-5c_2^4A_3\kappa^3c_1^3\\
&-15A_1c_2^4\kappa c_1+6A_1c_2^4\kappa^2c_1^2+5c_4\kappa c_1c_2+3\kappa^4c_1^4c_2^2c_3+c_4\kappa^3c_1^3c_2+68\kappa^2c_1^2c_2^2c_3-25c_3\kappa^3c_1^3c_2^2-2c_4c_2\\
&-78\kappa c_1c_2^2c_3-3\kappa^4c_1^4c_2^4+5\kappa c_1c_3^2+c_3^2\kappa^3c_1^3-2c_3^2-36c_2^4+32c_2^2c_3-66c_2^4\kappa^2c_1^2+80c_2^4\kappa c_1+24\kappa^3c_1^3c_2^4\\
&+9A_1c_2^2\kappa c_1c_3-3A_1c_2^2c_3\kappa^2c_1^2+21A_2c_2^2\kappa c_1c_3-27A_2c_2^2c_3\kappa^2c_1^2+15A_2c_2^2c_3\kappa^3c_1^3-3A_2c_2^2\kappa^4c_1^4c_3\\
&+15c_2^2A_3\kappa c_1c_3-12c_2^2A_3c_3\kappa^2c_1^2+3c_2^2A_3c_3\kappa^3c_1^3-6A_1c_2^2c_3-6A_2c_2^2c_3-6c_2^2A_3c_3+10A_1c_2^4+10c_2^4A_3\\
&+10A_2c_2^4)e_n^5+\cdots+O(e_n^9),\end{aligned}\tag{19}$$

$$\begin{aligned}f(z_n) &= c_1(-A_1c_2^2+6\kappa^2c_1^2c_2^2-c_3+2\kappa c_1c_3-10c_2^2\kappa c_1-c_3\kappa^2c_1^2+5c_2^2+A_1c_2^2\kappa c_1+2c_2^2A_3\kappa c_1-c_2^2A_3\kappa^2c_1^2\\
&+3A_2c_2^2\kappa c_1-3A_2c_2^2\kappa^2c_1^2+A_2c_2^2\kappa^3c_1^3-A_2c_2^2-\kappa^3c_1^3c_2^2-c_2^2A_3)c_2e_n^4+c_1(-4\kappa^2c_1^2c_3^2-4\kappa^2c_1^2c_4c_2\\
&-31A_2c_2^4\kappa c_1+36A_2c_2^4\kappa^2c_1^2-19A_2c_2^4\kappa^3c_1^3+4A_2c_2^4\kappa^4c_1^4-23c_2^4A_3\kappa c_1+18c_2^4A_3\kappa^2c_1^2-5c_2^4A_3\kappa^3c_1^3\\
&-15A_1c_2^4\kappa c_1+6A_1c_2^4\kappa^2c_1^2+5c_4\kappa c_1c_2+3\kappa^4c_1^4c_2^2c_3+c_4\kappa^3c_1^3c_2+68\kappa^2c_1^2c_2^2c_3-25c_3\kappa^3c_1^3c_2^2-2c_4c_2\\
&-78\kappa c_1c_2^2c_3-3\kappa^4c_1^4c_2^4+5\kappa c_1c_3^2+c_3^2\kappa^3c_1^3-2c_3^2-36c_2^4+32c_2^2c_3-66c_2^4\kappa^2c_1^2+80c_2^4\kappa c_1+24\kappa^3c_1^3c_2^4\\
&+9A_1c_2^2\kappa c_1c_3-3A_1c_2^2c_3\kappa^2c_1^2+21A_2c_2^2\kappa c_1c_3-27A_2c_2^2c_3\kappa^2c_1^2+15A_2c_2^2c_3\kappa^3c_1^3-3A_2c_2^2\kappa^4c_1^4c_3\\
&+15c_2^2A_3\kappa c_1c_3-12c_2^2A_3c_3\kappa^2c_1^2+3c_2^2A_3c_3\kappa^3c_1^3-6A_1c_2^2c_3-6A_2c_2^2c_3-6c_2^2A_3c_3\\
&+10A_1c_2^4+10c_2^4A_3+10A_2c_2^4)e_n^5+\cdots+O(e_n^9),\end{aligned}\tag{20}$$

$$\begin{aligned}\psi_n &= (2A_2c_2^3\kappa^3c_1^3-2\kappa^3c_1^3c_2^3+12\kappa^2c_1^2c_2^3-2\kappa^2c_1^2c_2c_3-6A_2c_2^3\kappa^2c_1^2+c_4\kappa^2c_1^2-2c_2^3A_3\kappa^2c_1^2-20\kappa c_1c_2^3\\
&+6A_2c_2^3\kappa c_1-2\kappa c_1c_4+2A_1c_2^3\kappa c_1+4c_2\kappa c_1c_3+4c_2^3A_3\kappa c_1+10c_2^3-2A_1c_2^3-2c_2^3A_3-2c_2c_3\\
&-2A_2c_2^3+c_4)c_1c_2e_n^4+\cdots+O(e_n^9).\end{aligned}\tag{21}$$

Finally, $H(s_1,s_2)$ has Taylor's expansion

$$H\left(\frac{f(z_n)}{f(x_n)},\frac{f(z_n)}{f(w_n)}\right)=1+B_1\left(\frac{f(z_n)}{f(x_n)}\right)^2+B_2\left(\frac{f(z_n)}{f(w_n)}\right)^2+B_3\left(\frac{f(z_n)}{f(x_n)}\right)\left(\frac{f(z_n)}{f(w_n)}\right)+O(s_1^3,s_2^3). \tag{22}$$

From (20) and (21), we deduced the following error equation which leads to the desired result

$$\begin{aligned}e_{n+1} &= c_2^2(-6\kappa^2c_1^2c_3c_4-10A_1c_2^5-10A_2c_2^5-10c_2^5A_3+6c_2\kappa^2c_1^2c_3^2+31\kappa^2c_1^2c_4c_2^2-4c_2\kappa^3c_1^3c_3^2\\
&+4c_3\kappa c_1c_4+4c_3c_4\kappa^3c_1^3+46c_3\kappa^3c_1^3c_2^3-23c_4\kappa^3c_1^3c_2^2+8c_4\kappa^4c_1^4c_2^2-c_4\kappa^4c_1^4c_3-4c_2\kappa c_1c_3^2-20\kappa c_1c_4c_2^2\\
&-c_3c_4-62c_3\kappa^2c_1^2c_2^3+\kappa^4c_1^4c_2c_3^2-16\kappa^4c_1^4c_2^3c_3-12\kappa^5c_1^5c_2^5+2\kappa^5c_1^5c_3c_2^3-\kappa^5c_1^5c_4c_2^2+6A_1c_2^3\kappa^2c_1^2c_3\\
&+3A_1c_2^2\kappa c_1c_4-6A_1c_2^3\kappa c_1c_3-2A_1c_2^3\kappa^3c_1^3c_3-3A_1c_2^2c_4\kappa^2c_1^2+A_1c_2^2c_4\kappa^3c_1^3-8c_2^3A_3\kappa^3c_1^3c_3+4c_2^2A_3\kappa c_1c_4\\
&-6c_2^2A_3\kappa^2c_1^2c_4+12c_2^3A_3\kappa^2c_1^2c_3+40c_2^3c_3\kappa c_1-A_1c_2^2c_4+2A_1c_2^3c_3-A_2c_2^2c_4+2A_2c_2^3c_3-c_2^2A_3c_4\\
&+2c_2^3A_3c_3+25c_2^5+2c_2^3A_3\kappa^4c_1^4c_3+4c_2^2A_3\kappa^3c_1^3c_4-c_2^2A_3\kappa^4c_1^4c_4-8c_2^3A_3\kappa c_1c_3-20A_2c_2^3\kappa^3c_1^3c_3\\
&+5A_2c_2^2\kappa c_1c_4-10A_2c_2^2\kappa^2c_1^2c_4+20A_2c_2^3\kappa^2c_1^2c_3+10A_2c_2^3\kappa^4c_1^4c_3+10A_2c_2^2\kappa^3c_1^3c_4-5A_2c_2^2\kappa^4c_1^4c_4\end{aligned}$$

$$
\begin{aligned}
&- 10A_2c_2^3\kappa c_1c_3 - 2A_2c_2^3\kappa^5c_1^5c_3 + A_2c_2^2\kappa^5c_1^5c_4 + c_2c_3^2 - 10c_2^3c_3 + 5c_2^2c_4 + 160\kappa^2c_1^2c_2^5 - 130\kappa^3c_1^3c_2^5\\
&- 100c_2^5\kappa c_1 + 56\kappa^4c_1^4c_2^5 - 32A_1c_2^5\kappa^2c_1^2 + 30A_1c_2^5\kappa c_1 + 14A_1c_2^5\kappa^3c_1^3 + 46c_2^5A_3\kappa^3c_1^3 - 62c_2^5A_3\kappa^2c_1^2\\
&- 16c_2^5A_3\kappa^4c_1^4 + 40c_2^5A_3\kappa c_1 + 108A_2c_2^5\kappa^3c_1^3 - 102A_2c_2^5\kappa^2c_1^2 - 62A_2c_2^5\kappa^4c_1^4 + 50A_2c_2^5\kappa c_1 + 18A_2c_2^5\kappa^5c_1^5\\
&+ A_1^2c_2^5 + A_2^2c_2^5 + c_2^5A_3^2 - 2A_1^2c_2^5\kappa c_1 - 2A_1c_2^5\kappa^4c_1^4 + A_1^2c_2^5\kappa^2c_1^2 - 4c_2^5A_3^2\kappa^3c_1^3 + 6c_2^5A_3^2\kappa^2c_1^2 - 4c_2^5A_3^2\kappa c_1\\
&+ 2c_2^5A_3\kappa^5c_1^5 + c_2^5A_3^2\kappa^4c_1^4 + 15A_2^2c_2^5\kappa^4c_1^4 - 20A_2^2c_2^5\kappa^3c_1^3 + 15A_2^2c_2^5\kappa^2c_1^2 - 6A_2^2c_2^5\kappa c_1 - 6A_2^2c_2^5\kappa^5c_1^5\\
&+ A_2^2c_2^5\kappa^6c_1^6 - 2A_2c_2^5\kappa^6c_1^6 - 8A_1c_2^5A_2\kappa^3c_1^3 + 12A_1c_2^5A_2\kappa^2c_1^2 + 6A_1c_2^5A_3\kappa^2c_1^2 - 8A_1c_2^5A_2\kappa c_1 - 6A_1c_2^5A_3\kappa c_1\\
&+ 2A_1c_2^5\kappa^4c_1^4A_2 - 2A_1c_2^5\kappa^3c_1^3A_3 + 10c_2^5A_3\kappa^4c_1^4A_2 - 20c_2^5A_3\kappa^3c_1^3A_2 + 20c_2^5A_3\kappa^2c_1^2A_2 - 10c_2^5A_3\kappa c_1A_2\\
&- 2c_2^5A_3\kappa^5c_1^5A_2 + 2A_1c_2^5A_3 + 2A_1c_2^5A_2 + 2A_2c_2^5A_3 + \kappa^6c_1^6c_2^5)e_n^8 + O(e_n^9). \qquad (23)
\end{aligned}
$$

□

It is clear that the considered family of numerical schemes requires four functional evaluations and attains optimal convergence order eight according to Kung and Traub conjecture which can be stated as follows [5]: if n is the total number of functional evaluations per iteration, then the optimal convergence order of the associated numerical procedure is 2^{n-1}.

4. Numerical Results

Definition 1. *The computational order of convergence [4], can be approximated by*

$$COC \approx \frac{ln|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}|}{ln|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}|}, \qquad (24)$$

where x_{n-1}, x_n and x_{n+1} are successive iterations closer to the root α of $f(x)=0$.

For the purpose of comparison between newly developed family and other derivative-free methods, a list of derivative-free methods for nonlinear equations is presented here.

4.1. The Kung-Traub Eighth-order Derivative-free Method (K-T)

The Kung-Traub eighth-order derivative-free method is discussed in [5, 6], and also considered in [7] is given as

$$
\begin{cases}
w_n &= x_n + \beta f(x_n),\\
y_n &= x_n - \left(\dfrac{\beta f(x_n)^2}{f(w_n)-f(x_n)}\right),\\
z_n &= y_n - \left(\dfrac{f(x_n)f(w_n)}{f(y_n)-f(x_n)}\right)\left[\dfrac{1}{f[w_n,x_n]} - \dfrac{1}{f[w_n,x_n]}\right],\\
x_{n+1} &= z_n - \left(\dfrac{f(w_n)f(x_n)f(y_n)}{f(z_n)-f(x_n)}\right)\\
&\quad \left\{\left(\dfrac{1}{f(z_n)-f(w_n)}\right)\left[\dfrac{1}{f[y_n,z_n]} - \dfrac{1}{f[w_n,y_n]}\right] - \left(\dfrac{1}{f(y_n)-f(x_n)}\right)\left[\dfrac{1}{f[w_n,y_n]} - \dfrac{1}{f[w_n,x_n]}\right]\right\}.
\end{cases} \qquad (25)
$$

4.2. R. Thukral $M1$, $M2$, $M3$ Methods

In 2011, R. Thukral [7] presented three variants of his proposed eighth-order three-point derivative-free method. Three members of the family called by author namely, $M1$, $M2$, and $M3$, are listed as

$$\phi_1 = \left(1 - \frac{f(y_n)}{f(w_n)}\right)^{-1}, \qquad (26)$$

$$\phi_2 = \left(1 + \frac{f(y_n)}{f(w_n)} + \left(\frac{f(y_n)}{f(w_n)}\right)^2\right), \qquad (27)$$

$$\phi_3 = \frac{f[x_n,w_n]}{f[w_n,y_n]}, \qquad (28)$$

and

$$
\begin{cases}
w_n &= x_n + \beta f(x_n),\\
y_n &= x_n - \left(\dfrac{\beta f(x_n)^2}{f(w_n) - f(x_n)}\right),\\
z_n &= y_n - \phi_k\left(\dfrac{f(y_n)}{f[x_n, y_n]}\right),\\
x_{n+1} &= z_n - \left(1 - \dfrac{f(z_n)}{f(w_n)}\right)^{-1}\left(1 - \dfrac{f(y_n)^3}{f(w_n)^2 f(x_n)}\right)\left(\dfrac{f[x_n, y_n] f(z_n)}{f[y_n, z_n] f[x_n, z_n]}\right),
\end{cases}
\tag{29}
$$

where $k = 1, 2, 3, \beta \in \Re^+$, ϕ_k are listed in (26)-(28). (29) is called $M1$, $M2$ and $M3$ for ϕ_1, ϕ_2 and ϕ_3 respectively.

4.3. *Petkovic et al. Type Methods*

In [7], author developed Petkovic type 1 (P1) and type 2 (P2) derivative-free methods for the comparison of numerical efficiency, (P1) and (P2) respectively, are written as

$$
\begin{cases}
w_n = x_n + \beta f(x_n),\\
y_n = x_n - \left(\dfrac{\beta f(x_n)^2}{f(w_n) - f(x_n)}\right),\\
z_n = y_n - \left(1 + \dfrac{f(y_n)}{f(w_n)} + \dfrac{f(y_n)}{f(x_n)}\right)\left[\dfrac{(w_n - x_n) f(y_n)}{f(w_n) - f(x_n)}\right],\\
x_{n+1} = z_n - \left(1 - \dfrac{f(z_n)}{f(w_n)}\right)^{-1}\\
\qquad \left(1 - \dfrac{2 f(y_n)^3}{f(w_n)^2 f(x_n)} - \dfrac{f(y_n)^3}{f(w_n) f(x_n)^2} - \left(\dfrac{f(y_n)}{f(w_n)}\right)^3\right)\left(\dfrac{f[x_n, y_n] f(z_n)}{f[y_n, z_n] f[x_n, z_n]}\right),
\end{cases}
\tag{30}
$$

and

$$
\begin{cases}
w_n = x_n + \beta f(x_n),\\
y_n = x_n - \left(\dfrac{\beta f(x_n)^2}{f(w_n) - f(x_n)}\right),\\
z_n = y_n - \left(\dfrac{1 + f(y_n) f(x_n)^{-1}}{1 - f(y_n) f(w_n)^{-1}}\right)\left(\dfrac{f(y_n)(w_n - x_n)}{f(w_n) - f(x_n)}\right),\\
x_{n+1} = z_n - \left(1 - \dfrac{f(z_n)}{f(w_n)}\right)^{-1}\left(1 - \dfrac{2 f(y_n)^3}{f(w_n)^2 f(x_n)} - \dfrac{f(y_n)^3}{f(w_n) f(x_n)^2}\right)\left(\dfrac{f[x_n, y_n] f(z_n)}{f[y_n, z_n] f[x_n, z_n]}\right).
\end{cases}
\tag{31}
$$

4.4. *Proposed family (L)*

We define the following weight functions:

$$G_1(t_1, t_2) = \frac{1}{1 - (t_1 + t_2) + \omega(t_1 + t_2)^2},\ \omega \in \Re, \tag{32}$$

$$G_2(t_1, t_2) = 1 + t_1 + t_2 + t_1^2 + 1.9 t_2^2 + 4.4 t_1 t_2, \tag{33}$$

$$H_1(s_1, s_2) = 1, \tag{34}$$

$$H_2(s_1, s_2) = \frac{1}{1 + s_1 s_2 + s_1^2 + s_2^2}, \tag{35}$$

$$H_3(s_1, s_2) = 1 + s_2^4 + s_2^6, \tag{36}$$

$$H_4(s_1, s_2) = 1 + s_1^2 + s_2^2 + 2 s_1 s_2, \tag{37}$$

$$H_5(s_1, s_2) = \frac{1}{1 - 20 s_1 s_2}, \tag{38}$$

Functions	Roots
$f_1(x) = exp(x)\ sin(x) + ln(1 + x^2)$	$\alpha = 0$
$f_2(x) = x^{15} + x^4 + 4x^2 - 15$	$\alpha = 1.148538...$
$f_3(x) = (x - 2)(x^{10} + x + 1)\ exp(-x - 1)$	$\alpha = 2$
$f_4(x) = exp(-x^2 + x + 2) - cos(x + 1) + x^3 + 1$	$\alpha = -1$
$f_5(x) = (x + 1)\ exp(sin(x)) - x^2\ exp(cos(x)) - 1$	$\alpha = 0$
$f_6(x) = sin(x)^2 - x^2 + 1$	$\alpha = 1.40449165...$
$f_7(x) = 10\ exp(-x^2) - 1$	$\alpha = 1.517427...$
$f_8(x) = (x^2 - 1)^{-1} - 1$	$\alpha = 1.414214...$
$f_9(x) = ln(x^2 + x + 2) - x + 1$	$\alpha = 4.15259074...$
$f_{10}(x) = cos(x)^2 - x/5$	$\alpha = 1.08598268...$
$f_{11}(x) = sin(x) - \frac{x}{2}$	$\alpha = 0$
$f_{12}(x) = x^{10} - 2x^3 - x + 1$	$\alpha = 0.591448093...$
$f_{13}(x) = exp(sin(x)) - x + 1$	$\alpha = 2.63066415...$

$(f_n(x), x_0)$	L	K-T	M1	M2	M3	P1	P2
f_1, 0.25	(L1) 6.38e-247	3.14e-136	1.69e-141	7.43e-142	1.69e-141	3.20e-113	8.98e-120
f_2, 1.1	(L1) 1.2376e-652	3.72e-61	3.44e-62	3.44e-62	3.44e-62	2.68e-7	2.67e-7
f_3, 2.1	(L1) 1.057e-422	1.91e-60	1.49e-60	1.49e-60	1.49e-60	7.71e-8	7.56e-8
f_4, -0.5	(L1) 2.952e-383	5.11e-362	1.92e-362	1.93e-362	1.92e-362	9.99e-367	8.78e-366
f_5, 0.25	(L1) 2.336e-407	4.13e-328	6.52e-326	9.47e-326	6.52e-326	1.98e-322	2.56e-332
f_6, 1.2	(L8) 1.719e-421	1.00e-327	4.58e-341	7.57e-344	4.58e-341	1.79e-381	1.72e-405
f_7, 2	(L2) 7.264e-238	5.19e-88	1.24e-120	6.40e-124	1.24e-120	1.51e-187	6.79e-228
f_8, 1.7	(L3) 1.429e-234	1.23e-113	1.74e-171	5.45e-188	1.74e-171	5.96e-211	4.84e-167
f_9, 4.4	(L4) 2.504e-997	1.15e-928	4.52e-942	1.27e-965	4.52e-941	6.15e-904	4.11e-937
f_{10}, 1.5	(L5) 2.81e-305	7.19e-303	5.07e-284	1.84e-245	5.07e-285	4.91e244	1.78e-275
f_{11}, 0.25	(L6) 2.35e-1143	3.65e-782	1.00e-819	4.98e-823	1.00e-819	5.13e-794	5.13e-812
f_{12}, 0.25	(L6) 7.86e-318	2.03e-256	5.65e256	1.82e-254	5.65e-256	1.07e-264	6.31e-268
f_{13}, 2.0	(L7) 2.54e-436	2.63e-396	1.94e-378	5.1e-378	1.94e-378	8.70e-380	6.80e-379
			(COC)				
f_1	(L1) 7.9999	7.9986	7.9995	7.9998	7.9995	7.9958	7.9978
f_2	(L1) 8.0000	7.8671	7.9371	7.9371	7.9371	3.2715	3.2731
f_3	(L1) 7.9999	7.8660	7.9047	7.9047	7.9047	4.2595	4.2675
f_4	(L1) 8.0000	7.9905	7.9905	7.9905	7.9905	7.9907	7.9907
f_5	(L1) 8.0000	7.9884	7.9882	7.9882	7.9882	8.0000	8.0000
f_6	(L8) 8.0000	8.000	8.0000	8.0000	8.0000	8.0000	8.0000
f_7	(L2) 7.9999	8.0097	8.0025	8.0018	8.0025	8.0005	8.0001
f_8	(L3) 8.0000	8.0027	8.0004	8.0002	8.0004	8.0000	8.0002
f_9	(L4) 8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
f_{10}	(L5) 8.0000	8.0002	7.9845	7.9797	7.9845	8.0003	7.9833
f_{11}	(L6) 11.000	10.996	10.996	10.996	10.996	10.996	10.996
f_{12}	(L6) 8.0000	8.0000	7.9809	7.9807	7.9809	7.9822	8.0000
f_{13}	(L7) 7.9999	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000

Table 1: Numerical comparison between three-point derivative-free methods

where t_i and s_i are defined in (10). Further we give names to methods for the purpose of simplicity as follows

$$\begin{cases} L1 = (G_1,\ H_1,\ \omega = +0.01,\ \kappa = 0.01),\ L2 = (G_1,\ H_1,\ \omega = -0.022,\ \kappa = 0.01), \\ L3 = (G_1,\ H_1,\ \omega = -0.001,\ \kappa = 0.01),\ L4 = (G_2,\ H_1,\ \omega = +0.01,\ \kappa = 0.01), \\ L5 = (G_1,\ H_3,\ \omega = -0.01,\ \kappa = 0.01),\ L6 = (G_1,\ H_2,\ \omega = +0.01,\ \kappa = 0.01), \\ L7 = (G_1,\ H_4,\ \omega = +0.01,\ \kappa = 0.01),\ L8 = (G_1,\ H_5,\ \omega = +0.01,\ \kappa = 0.01). \end{cases} \tag{39}$$

A set of thirteen nonlinear equations is used for numerical computations from [7], in Table 1. All the families in the numerical implementation are derivative-free and use four function evaluations to get the order of convergence eight . For all methods, 12 (TNFE) total number of function evaluations are used, and absolute error ($|x_n - \alpha|$) is displayed. Computational order of convergence is calculated according to (24) for the method. All numerical values for methods K-T, M1, M2, M3, P1, P2 are taken from [7].

5. Conclusion

In this note, we have presented a family of eighth-order derivative-free methods. The proper selection of weight functions showed a reasonable reduction in error as compared to other referenced derivative-free methods. It is obvious that constructed family has broad choice for the weight function in the third and fourth step of the method. The true essence of the family is hidden in the construction of interpolation polynomial for the approximation of $f'(z)$ and weight functions make it more flexible to get higher performance and efficiency.

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